Probability Trees:

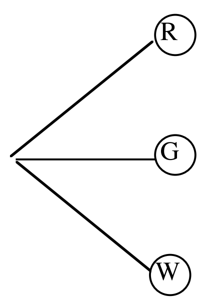
Many probability problems can be simplified by using a device called a *probability tree*. This is especially (but not exclusively) true in situations where actions are taken or decisions are made sequentially. Probability trees provide a systematic method of generating the elements of a suitable sample space and determining their probabilities.

Constructing the Tree for a Sequential Process:

Probability trees may be shown growing from left to right or from top to bottom. The *root* of the tree corresponds to the starting point of the process. Line segments called *branches* connect the root to *nodes* representing the different outcomes that are possible at the first stage of the process. Each of those stage 1 nodes is connected to nodes representing the possible outcomes at the next stage, and the process continues until all stages are completed.

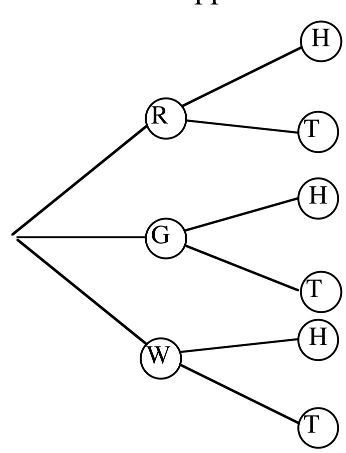
Consider and example where you have three (3) red balls, two (2) green balls and two (1) white ball in an urn. One ball is chosen randomly from the urn. If a red or white ball is chosen, a fair coin is flipped once. If the ball is green, the coin is flipped twice. We can construct a tree to enumerate the outcomes (the elements of the sample space) for this random process In the tree, the letters R, G and W represent the colors red, green and white, and the letters H and T represent heads and tails, respectively

Stage 1: A ball is chosen and its color is noted:



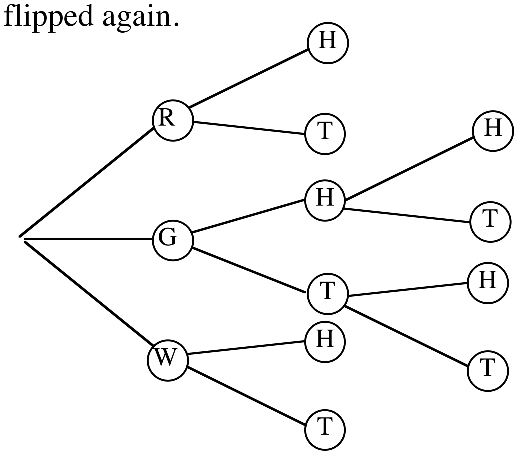
Notice that there are three branches because there are three possible colors.

Stage 2: The coin is flipped and the result (head or tail) is noted:



From this portion of the tree, we can determine all of the possible sequences of events in the first two stages: RH (red followed by head), RT (red followed by tail), GH (green followed by head), GT (green followed by tail), WH (white followed by head), and WT (white followed by tail).

Stage 3: If the ball chosen at stage1 was green, The coin is flipped a second time and the result is noted. If the ball chosen at stage 1 was red or white, the coin is not flipped again.



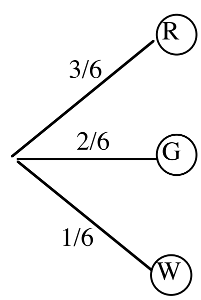
Each path from the starting point to a node at the other end of the tree represents one outcome of the complete process: RH, RT, GHH, GHT, GTH, GTT, WH, WT.

Notice that some information was not retained. There are actually six balls in the urn. We could have numbered the red balls from 1 to 3 and the green balls from 1 to 2 in order to tell them apart and then started the tree with six branches. We did not bother to do this because the color was all that we needed to know about the ball to determine the steps taken at later stages. This simplifies the tree, making it much easier to draw. In general, keep only as much information (and as many branches) as you need to represent the information that is of interest at the current stage and later stages.

Labeling the Branches with Conditional Probabilities:

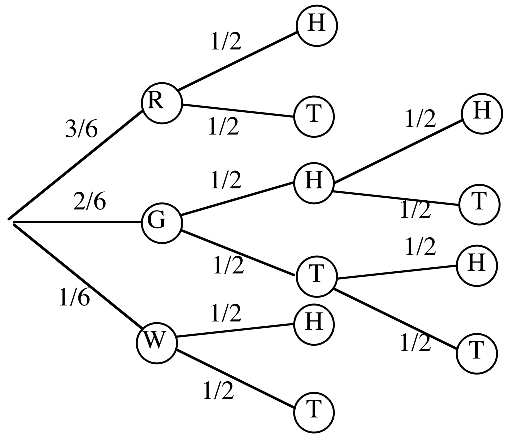
It is clear that the eight outcomes in our example are not equally likely. There are more likely to draw a red ball than either other color, and a green ball is followed by two flips rather than one. Fortunately, probability trees provide an easy way to handle this complication.

Branch Probabilities at Stage 1: Place the probabilities of the stage 1 outcomes on the branches that lead from the starting point to the outcome



Here we are assuming that each of the six balls is equally likely to be chosen Consequently, the probability of choosing a particular color is just the proportion of the six balls of that color.

Branch Probabilities at All Later Stages: Place *conditional* probabilities on the branches at later stages: the condition is the history of choices represented by the path from the start to the beginning of the branch.



Since the coin is fair, the probability of flipping a head and the probability of flipping a tail are both 1/2 on each flip. In this case, the probabilities do not depend on what has happened before.

Using the Tree to Calculate Probabilities

Probabilities of Individual Outcomes:

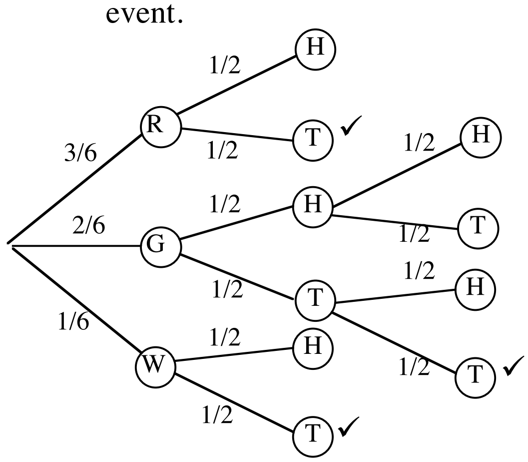
To determine the probability of an outcome, multiply the probabilities along its path. (This is a consequence of the *Multiplicative Law of Probability*.) For example, the probability of drawing a red ball followed by a tail is (3/6)(1/2) = 1/4, and the probability of drawing a green ball followed by two heads is (2/6)(1/2)(1/2) = 1/12.

Probabilities of Compound Events:

Since different paths in a probability tree represent mutually exclusive events, we can add their probabilities without worrying about any overlap to find the probability of a compound event. (This is a consequence of the Additive Law of Probability.)

To determine the probability of a compound event:

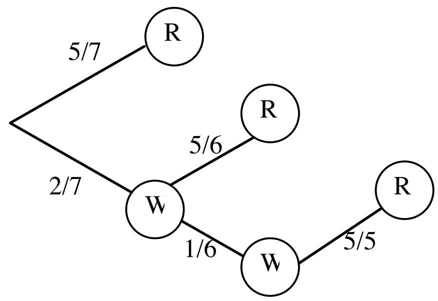
(1) Check off the paths in the tree that satisfy the specified condition(s).
(2) Find the probability of each selected path by multiplying the probabilities along its branches.
(3) Add the probabilities of all of the selected paths to find the probability of the event.



For example, to determine the probability that you get no heads in the process described above:

(1) Find all of the paths in the tree which have no heads (they are checked.)
(2) and (3) Determine the probabilities of the paths and add them together:
(3/6)(1/2) + (2/6)(1/2)(1/2) + (1/6)(1/2)
= 1/4 + 1/12 + 1/12 = 5/12

Consider another example where an urn contains 2 white balls and 5 red balls. You are to randomly select balls from this box without replacement, until you get a red ball. The following probability tree represents this experiment:

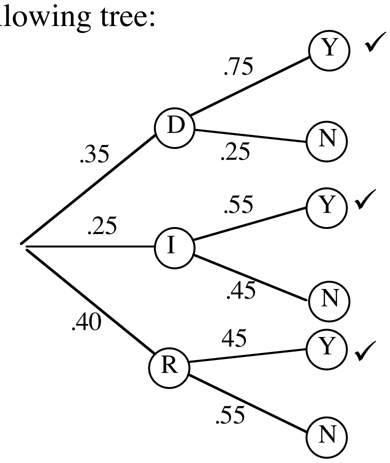


Notice that the history of events does affect the (conditional) probabilities of events at later stages in this case. Since we are drawing without replacement, the number and color mixture of the balls changes from draw to draw.

Probability Trees and Non-Sequential Processes:

Although the progressive structure of trees seems most naturally suited for experiments conducted in stages where outcomes are sequences of observations, probability trees may also be useful in other situations. Consider the following example where the first stage breaks the population into subgroups that are more easily treated separately.

Suppose 75% of Democrats and 55% and 55% of Independents favor a certain ballot proposal, but only 45% of Republicans favor it. What is the probability that a randomly selected voter will favor the proposal if 35% of voters are Democrats, 25% are Independents and 40% are Republicans? We can use a probability tree to answer this question. Let D, I and R represent the party affiliation of the selected voter and Y(yes) and N (no) whether the voter favors or does not favor the proposal. We can construct the following tree:



Although we are making only one selection, we can imagine that we are making two: first decide whether to interview a Democrat, Republican of Independent, then choose a voter randomly from that subgroup.

The probability that the voter will favor the proposal can be calculated easily:
$(.35)(.75) + (.25)(.55) + (.40)(.45) = .58$

This technique is very useful for extending detailed information about sub-populations to the whole population.